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Infinite friezes of cluster algebras from surfaces

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Abstract. Originally studied by Conway and Coxeter, friezes appeared in various recreational mathematics publications in the 1970s. More recently, in 2015, Baur, Parsons, and Tschabold constructed periodic infinite friezes and related them to matching numbers in the once-punctured disk and annulus. In this paper, we study such infinite friezes with an eye towards cluster algebras of type D and affine A, respectively. By examining infinite friezes with Laurent polynomials entries, we discover new symmetries and formulas relating the entries of this frieze to one another.

Keywords: cluster algebra, Conway–Coxeter frieze, triangulation, marked surface

1 Introduction

A Conway-Coxeter *frieze* $\mathcal{F} = \{\mathcal{F}_{ij}\}_{i \leq j}$ is an array of rows (arranged and indexed as in Figure 1) such that $\mathcal{F}_{i,i} = 0$ and $\mathcal{F}_{i,i+1} = 1$, and, for every diamond

$$\begin{matrix} & c & \\ a & & b \\ & d & \end{matrix}$$

of entries in the frieze, the equation $ab - cd = 1$ is satisfied.

We say a frieze is *finite* if it is bounded above and below by a row of 1s. In the 70s, Conway and Coxeter showed that finite friezes with positive integer entries are in bijection with triangulations of polygons [10, 9]. Given a triangulation T of a polygon, each entry of the second row of the corresponding frieze is the number of triangles adjacent to a vertex. Broline, Crowe, and Isaacs further studied this in [6] and found that every entry in such a frieze corresponds to a diagonal (see Figure 2). To any diagonal, they associate a set of vertices $v_{i_1}, \ldots, v_{i_r}$ (those lying to the right) and then match these to a *BCI r-tuple* $(t_1, \ldots, t_r)$ of pairwise-distinct triangles in T, such that t_j is incident to vertex v_{i_j}. For example, in Figure 3, the diagonal from vertex v_5 to vertex v_3 is associated to the vertices v_1 and v_2. There are exactly two BCI 2-tuples corresponding to v_1 and v_2.

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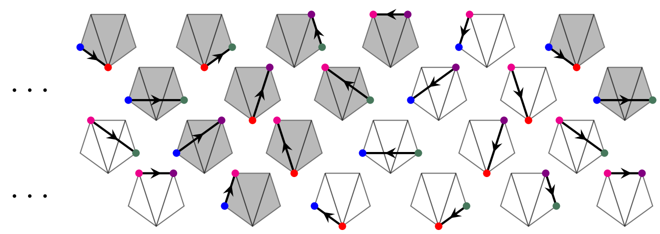
$$
\begin{array}{ccccccccccccc}
 & 0 & & 0 & & 0 & & 0 & & 0 & & & \\
\cdots & & 1 & & 1 & & 1 & & 1 & & 1 & & \cdots \\
 & & & \mathcal{F}_{-1,1} & & \mathcal{F}_{0,2} & & \mathcal{F}_{1,3} & & \mathcal{F}_{2,4} & & \mathcal{F}_{3,5} & \\
 & & \cdots & & \mathcal{F}_{-1,2} & & \mathcal{F}_{0,3} & & \mathcal{F}_{1,4} & & \mathcal{F}_{2,5} & & \mathcal{F}_{3,6} \quad \cdots \\
 & & & & & \mathcal{F}_{-1,3} & & \mathcal{F}_{0,4} & & \mathcal{F}_{1,5} & & \mathcal{F}_{2,6} & \mathcal{F}_{3,7} \\
 & & & & & & & \ddots & & & & \ddots &
\end{array}
$$

Figure 1: Arrangement and indices for frieze entries.



		1	**1**	**1**	**1**	**1**	**1**	**1** ⋯
Row 2	⋯	**3**	**1**	**2**	**2**	**1**	**3**	**1**
		2	**2**	**1**	**3**	**1**	**2**	**2** ⋯
	⋯	**1**	**1**	**1**	**1**	**1**	**1**	**1**

Figure 2: Diagonals of a polygon correspond to entries of a finite frieze.

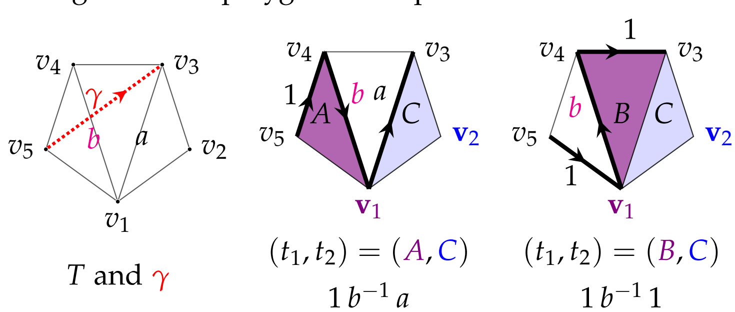


Figure 3: The BCI 2-tuples for γ which match vertices v_1 & v_2 to their adjacent triangles, and the corresponding trails whose weights sum up to the expansion $x_\gamma = \frac{a}{b} + \frac{1}{b}$.

More recently, Caldero and Chapoton in [7] showed that finite frieze patterns appear in the context of Fomin–Zelevinsky cluster algebras [12] of type A. Carroll and Price in [8] gave an expansion formula for cluster variables of type A in terms of BCI tuples (see [14, Appendix A] and Figure 3). Enumerating BCI tuples is equivalent to counting perfect matchings in a bipartite graph whose nodes are the triangles and vertices of (respectively a snake graph associated to) a triangulation; see Sec. 2 (respectively Sec. 4)

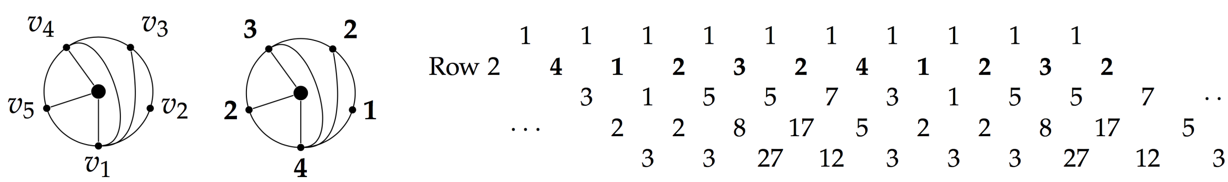


Figure 4: First 5 rows of the infinite frieze from a pentagon triangulation.

of [17].

A frieze is said to be *infinite* if it is not bounded below by a row of 1s. Infinite friezes of positive integers arising from once-punctured disks were introduced in [21] by Tschabold. Given an ideal triangulation T (in the sense of [11]) of a once-punctured disk with n marked boundary vertices labeled $v_1, v_2, \ldots, v_n$ counterclockwise around the boundary, we can count the number of BCI tuples in a similar way, see Figure 4.

In [3], Baur, Parsons, and Tschabold went further and gave a complete characterization of infinite friezes of positive integers via triangulations of quotients of an infinite strip in the plane. In this classification, periodic friezes arise from triangulations of the annulus or of the once-punctured disk (which can be thought of as a quotient of the infinite strip), see Figure 5. An infinite frieze is said to be of type D or type $\tilde{A}$, if it arises from a once-punctured disk or annulus, respectively. Related work on friezes of type D and $\tilde{A}$ include [18, 2, 1, 5, 19, 4, 13].

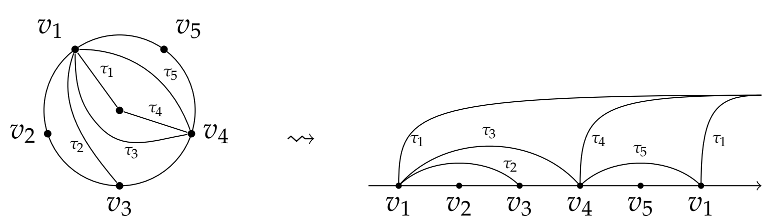


Figure 5: Triangulation of a once-punctured pentagon drawn as an asymptotic triangulation (see [3, Lemma 3.6]).

This is an extended abstract of [14], where we study infinite friezes whose entries are Laurent polynomials (as opposed to positive integers). In Section 2, we give the necessary background. In Section 3.1, we construct an infinite frieze of Laurent polynomials associated to *generalized peripheral arcs*. In Section 3.2, we introduce *complementary arcs*, which are arcs between the same two vertices in a surface, but of alternate direction. We use these complementary arcs to describe *progressions* of arcs in the frieze (Theorem 3.4). In Section 3.3, we show that *growth coefficients* (as defined in [4]) of the frieze are equal to Laurent polynomials corresponding to certain curves called *bracelets* in the surface. We state further combinatorial results in Sections 3.4 and 3.5.

2 Background

2.1 Cluster algebras from surfaces

We provide a brief background on cluster algebras arising from marked surfaces (S, M), following Fomin, Shapiro, and Thurston [11]. Let S be either an annulus or a disk, and M a non-empty, finite set of marked points in the closure of S, such that there is at least one marked point on each boundary component of S. The interior marked points in S are called *punctures*.

A *generalized ordinary arc* γ in (S, M) is a curve in S, considered up to isotopy, such that: (1) the endpoints of γ are in M, (2) the interior of γ is disjoint from M and from the boundary of S, (3) γ does not cut out an unpunctured monogon or bigon. Generalized arcs are allowed to intersect themselves a finite number of times. We consider these up to isotopy of immersed arcs, that is, allowing Reidemeister moves of types II and III but not of type I. In particular, an isotopy cannot remove a contractible kink (see Figure 15, bottom) from a generalized arc. If an arc intersects itself in its interior, we say that the arc has a *self-crossing*.

A *boundary edge* is a curve that connects two marked points and lies entirely on the boundary of S without passing through a third marked point. A generalized arc γ is called *peripheral* on a single boundary component Bd of S if: (1) both its endpoints (or its unique endpoint in the case of a loop) are on Bd, and (2) γ is isotopic to a concatenation of two or more boundary edges of a boundary component Bd. Our convention is to choose the orientation of γ so that Bd is to the right of γ when looking from above.

An *ordinary arc* γ is a generalized ordinary arc which has no self-crossing. We say that two ordinary arcs α, β are *compatible* if there exist representatives α', β' in their respective isotopy classes such that α' and β' do not intersect in the interior of S. An *ideal triangulation* T is a maximal (by inclusion) collection of distinct, pairwise compatible ordinary arcs (see Figure 5, left).

Due to [11, Thm. 7.11], we can associate a *signed adjacency matrix* B_T, and hence a cluster algebra, to T. The ordinary arcs τ of (S, M) correspond to cluster variables and products of cluster variables, denoted by x_τ or $x(\tau)$.

2.2 Laurent polynomials associated to generalized arcs and closed loops

In [16, 15], Schiffler, Williams, and the second author associated to a generalized arc (respectively, closed loop) γ and an ideal triangulation T a Laurent polynomial X^T_γ which is a weighted sum over perfect matchings of a planar *snake graph* (respectively, a *band* on a Möbius strip or annulus) $G_{T,\gamma}$. Put simply, such graphs are made out of gluing squares together, with one such square for each arc of T crossed by γ; see Figure 6, right. A *perfect matching* of a graph G is a subset P of the edges of G such that each vertex of G

is incident to exactly one edge of P. If G is a snake or band graph, and the edges of a perfect matching P of G are labeled $\tau_{j_1}, \ldots, \tau_{j_r}$, then we define the *weight* $x(P)$ of P to be $x_{\tau_{j_1}} \cdots x_{\tau_{j_r}}$. If τ is a boundary segment, we let $x_\tau := 1$.

Definition 2.1 ([15, Def. 3.12])**.** *Let T be an ideal triangulation, $\mathcal{A}$ the cluster algebra associated to B_T, and γ be a generalized arc. We define a Laurent polynomial which lies in $\mathcal{A}$.*

1. *If γ cuts out a contractible monogon, then X_γ^T is equal to zero.*
2. *If γ has a contractible kink, let $\overline{\gamma}$ denote the corresponding generalized arc with this kink removed, and define $X_\gamma^T := (-1)X_{\overline{\gamma}}^T$.*
3. *Otherwise, let $\tau_{i_1}, \tau_{i_2}, \ldots, \tau_{i_d}$ be the sequence of arcs in T which γ crosses. Define*

$$X_\gamma^T := \frac{1}{\tau_{i_1}\,\tau_{i_2}\,\ldots\,\tau_{i_d}} \sum_P x(P),$$

where the sum is over all perfect matchings P of $G_{T,\gamma}$.

Theorem 2.2 ([16, Thm 4.10])**.** *When γ is an ordinary arc with no self-crossings, X_γ^T is equal to the Laurent expansion of the cluster variable x_γ with respect to T.*

Example 2.3. *The snake graph $G_{T,\gamma}$ associated to the generalized arc γ in Figure 6 has* 11 *perfect matchings. Following Definition 2.1, we compute*

$$X_\gamma^T = \frac{x_0x_1x_4 + 2x_1x_3x_4 + 2x_0^2 + 4x_0x_3 + 2x_3^2}{x_0x_1x_4}$$

by specializing $x_\tau = 1$ for each boundary edge τ.

A Laurent polynomial X_ζ^T is associated to any closed loop ζ by a similar formula, see [15, Def. 3.14]. A closed loop obtained by following a (non-contractible, non-self-crossing, kink-free) loop k times, and thus creating $k-1$ self-crossings, is called a k-*bracelet* and is denoted by $Brac_k$, see Figure 7.

For the rest of the paper, we will use the notation x_γ or $x(\gamma)$ to denote the Laurent polynomial corresponding to γ, where γ is a generalized arc or loop.

3 Results

3.1 Infinite friezes of cluster algebra elements

Theorem 3.1. *Let T be an ideal triangulation of a once-punctured disk or an annulus. Let Bd be a boundary component with n marked points, where $n \geq 2$. Then the Laurent polynomials corresponding to generalized peripheral arcs on Bd form an infinite frieze pattern.*

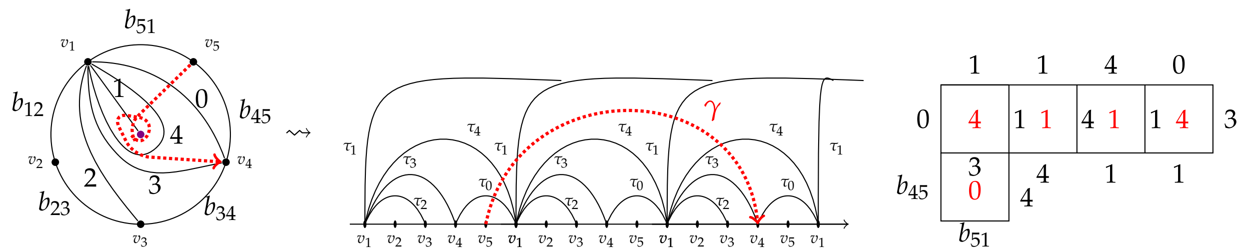


Figure 6: Left: An ideal triangulation T and a generalized arc γ of a once-punctured disk. Center: drawn on a strip. Right: Snake graph $G_{T,\gamma}$.

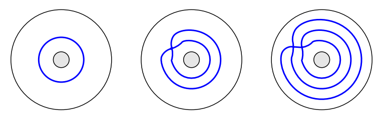


Figure 7: Bracelets $Brac_1$, $Brac_2$, and $Brac_3$.

We prove this theorem by applying skein relations, as illustrated in Figure 8. Further, we lift arcs from a once-punctured disk (or annulus) to a covering space given by the infinite strip. Given a triangulation of the infinite strip with marked points on a boundary ∂, the peripheral arc $\gamma(i, j)$ from i to j on ∂ corresponds to the (i, j)-th entry in the infinite frieze pattern arising from this triangulation. See Figures 9 and 10.

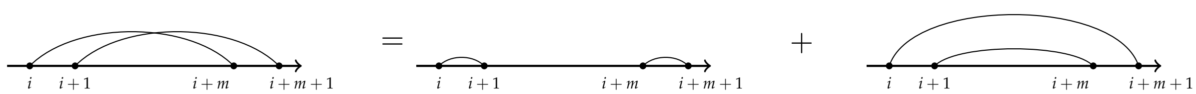


Figure 8: Applying skein relations to prove Theorem 3.1

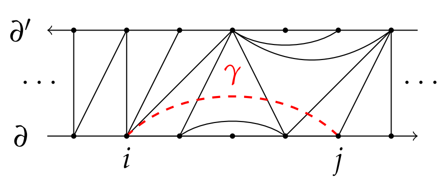


Figure 9: Triangulation of a strip and an arc $\gamma(i, j)$ from i to j.

3.2 Complementary arcs and progression formulas

In this section, we present formulas governing relations among the Laurent polynomial entries of the frieze of Theorem 3.1. These generalize the relations given in [4, Thm. 2.5].

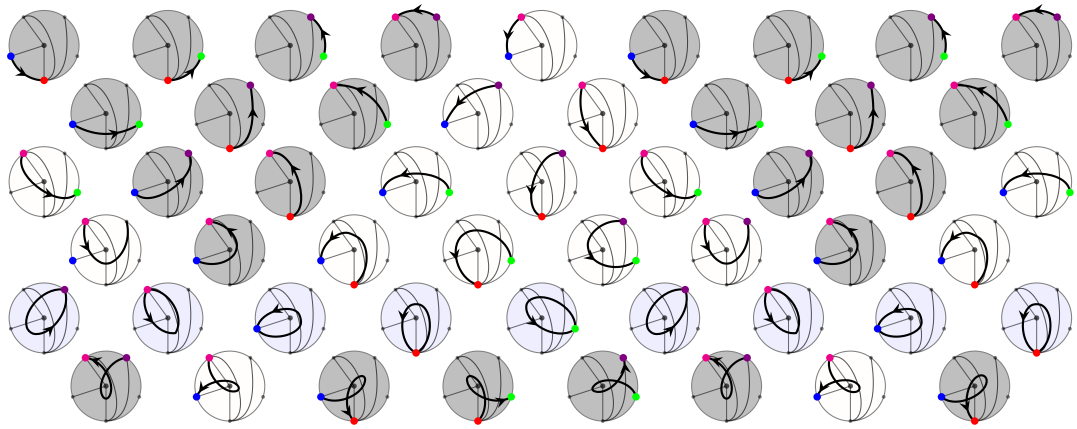


Figure 10: The first six rows of an infinite frieze of elements of the cluster algebra corresponding to peripheral arcs of a punctured disk.

For $1 \leq i, j \leq n$ and $k = 1, 2, \ldots$, we let $\gamma_k(i, j)$ denote the generalized peripheral arc that lifts to the covering by the strip as follows:

$$\gamma_k(i,j) = \begin{cases} \gamma\,(i, j + (k-1)n) & \text{if } i < j \\ \gamma\,(i, j + kn) & \text{if } i \geq j \end{cases}.$$

That is, $\gamma_k(i, j)$ is the generalized peripheral arc that starts at marked point i and finishes at the marked point j (possibly with $i = j$) with $(k-1)$ self-crossings such that the boundary Bd is to the right of the curve as we trace it.

Definition 3.2 (Complementary arc)**.** *Using the above shorthand notation, we define the arc complementary to $\gamma_k = \gamma_k(i, j)$ as*

$$\gamma_k(i,j)^C = \begin{cases} \gamma\,(j, i + kn) & \text{if } i < j \\ \gamma\,(j, i + (k-1)n) & \text{if } i \geq j \end{cases}.$$

Remark 3.3. *When $i \neq j$, the complementary arc γ_k^C to $\gamma_k = \gamma_k(i, j)$ is the generalized arc starting at j and finishing at i and retaining $(k-1)$ self-crossings while following the orientation of the surface. See Figure 11. In this case, $(\gamma_k^C)^C = \gamma_k$. On the other hand, when $i = j$, complementation is non-involutive and simply decreases the number of self-intersections by one.*

Theorem 3.4 (Progression formulas)**.** *Let γ_1 be a peripheral arc or a boundary edge of (S, M) starting and finishing at points i and j. For $k = 2, 3, \ldots$ and $1 \leq m \leq k-1$, we have*

$$x(\gamma_k) = x(\gamma_m)\, x(Brac_{k-m}) + x(\gamma^C_{k-2m+1}). \tag{3.1}$$

For $r \geq 0$, γ^C_{-r} is defined to be the curve γ_{r+1} with a kink, so that $x(\gamma^C_{-r}) = -x(\gamma_{r+1})$.

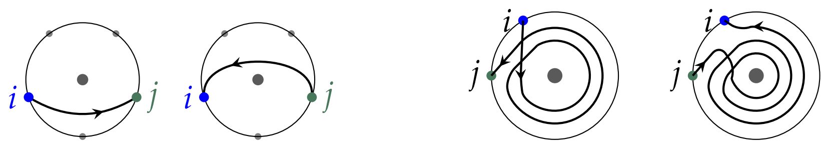


Figure 11: Examples of involutive complementary arcs γ_1, γ_1^C and γ_3, γ_3^C.

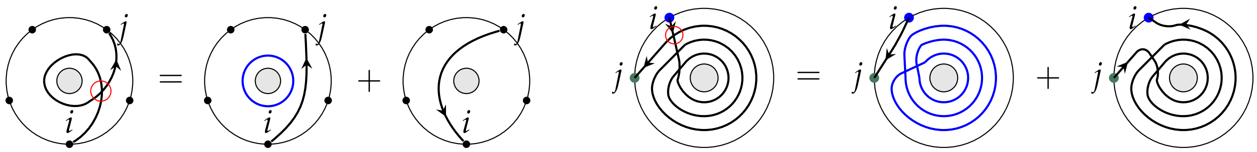


Figure 12: Case $m = 1$ for the progression formula (Theorem 3.4). Left: $x(\gamma_2) = x(\gamma_1)x(Brac_1) + x(\gamma_1^C)$. Right: $x(\gamma_4) = x(\gamma_1)x(Brac_3) + x(\gamma_3^C)$.

Remark 3.5. *In the above theorem, when $m = 1$ (see Figure 12) and $m = k-1$, we have*

$$x(\gamma_k) = x(\gamma_1)\, x(Brac_{k-1}) + x(\gamma_{k-1}^C) \text{ and } x(\gamma_k) = x(\gamma_{k-1})\, x(Brac_1) - x(\gamma_{k-2}). \qquad (3.2)$$

Compare (3.2), right, with [4, Thm. 2.5].

We give a sketch of our proof of Theorem 3.4. Let $\gamma_k := \gamma_k(i,j)$. We draw γ_k so that it first closely follows the other boundary (or the puncture) and then spirals out. In the covering via the infinite horizontal strip, we draw the lower boundary Bd so that i is drawn to the left of j in each frame. Each representative of γ_k is drawn starting from a vertex labeled i at a frame Reg_0. We go north, passing through all of the $(k-1)$ crossings. We then turn southeast and finish at a vertex labeled j, which is located in the frame $k-1$ frames (respectively, k frames) east of Reg_0 if $i \neq j$ (respectively, if $i = j$). See Figure 13.

We order the crossings of γ_k so that the first crossing is the one closest to Bd and the $(k-1)$-th crossing is the one furthest away from Bd. Resolving each representative of the m-th crossing in each frame, we get γ_m and $Brac_{k-m}$ (see Figure 14) and the curve γ_{k-2m+1}^C (see Figure 15), which correspond to the first and second summands of (3.1), respectively.

3.3 Bracelets and growth coefficients

According to [4, Thm. 2.2], for an n-periodic infinite frieze of positive integers, the difference between the entries in rows $(nk+1)$ & $(nk-1)$ and the same column is a constant (see Figure 16). These differences are also constant in our infinite friezes of Laurent polynomials, and we give geometric interpretations to these differences.

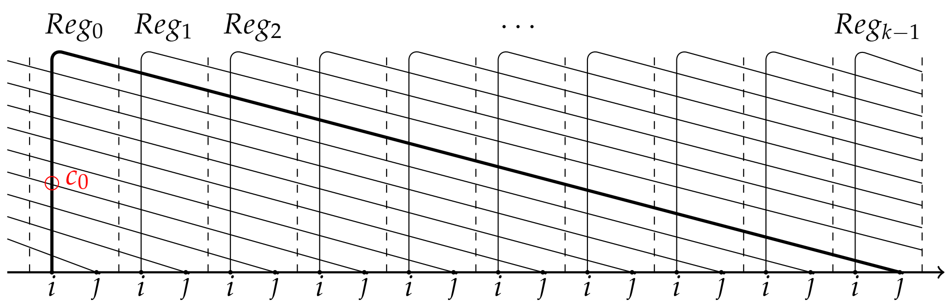


Figure 13: Lift of γ_k for $k = 10$, $m = 4$ drawn on the strip.

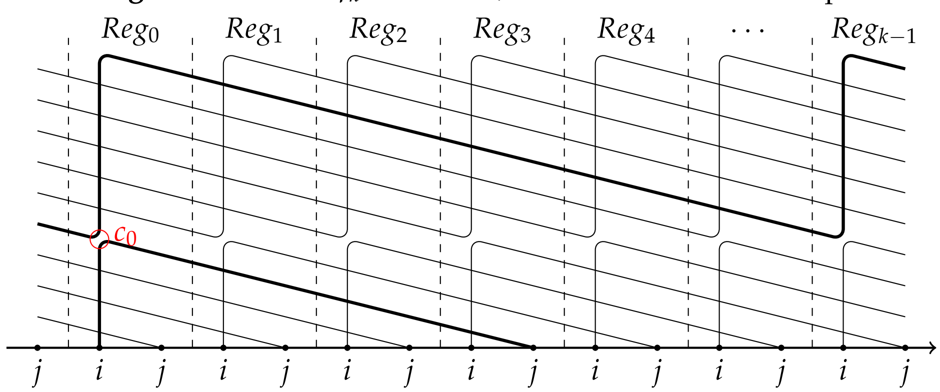


Figure 14: Lifts of γ_m and $Brac_{k-m}$ for $k = 10$, $m = 4$ drawn on the strip.

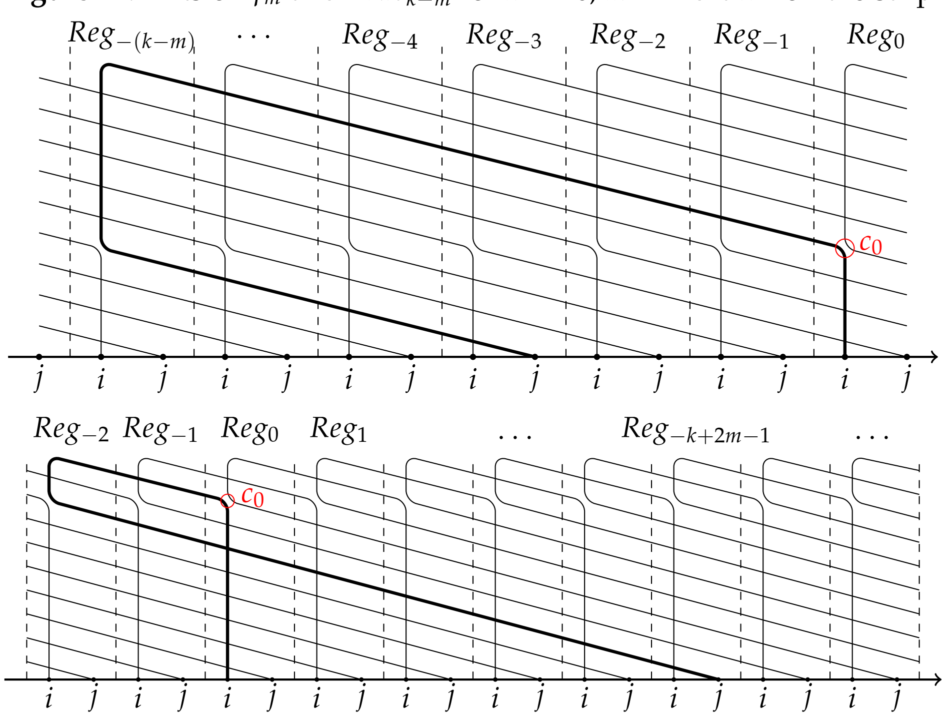


Figure 15: Lift of γ^C_{k-2m+1} for $k = 10$ drawn on the strip. Top: $m = 4$, Bottom: $m = 8$.

Proposition 3.6. *Let $\mathcal{F} = \{\mathcal{F}_{i,j}\}$ be an n-periodic frieze as described in Section 3.1. For each $k \geq 1$,*

$$x(Brac_k) = \mathcal{F}_{i,i+1+kn} - \mathcal{F}_{i+1,i+kn} \text{ for all } i \in \mathbb{Z}.$$

Following [4, Def. 2.3], for $k \geq 0$, we define the *kth growth coefficient* for $\mathcal{F}$ to be $s_0 := 2$, and $s_k := \mathcal{F}_{i,i+1+kn} - \mathcal{F}_{i+1,i+kn}$, otherwise. We say that *level k* of a frieze consists of the entries of the frieze indexed by $(i, i + (k-1)n + j)$ where $j = 1, \ldots, n$. Note that s_k measures the difference between entries in the first row of the $(k+1)$st level and the penultimate row of the kth level. Also, per Proposition 3.6, $s_k = x(\mathrm{Brac}_k)$ whenever $k \geq 1$, so we can use the two terms interchangeably.

Given a triangulation of an annulus, we get two different friezes corresponding to the outer and inner boundaries. We see that their growth coefficients s_k coincide since Brac_k is defined independently of the choice of the boundary of an annulus. This agrees with [4, Thm. 3.4].

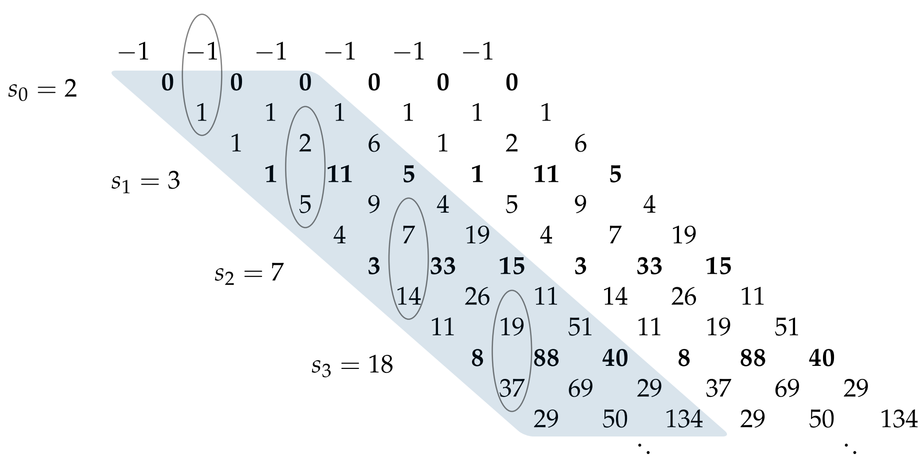


Figure 16: Growth coefficients in a frieze of type $\tilde{A}$.

3.4 Differences from complement symmetry

We consider the difference between frieze entries associated to complementary arcs. For the once-punctured disk, this difference is constant across all levels and is determined only by the endpoints of each arc.

Proposition 3.7. *Let $\mathcal{F}$ be a frieze coming from a triangulation of a once-punctured disk or annulus. Let $\gamma_1 = \gamma$ be an ordinary arc from i to j (possibly $i = j$) or a boundary edge from i to $i+1$. Define $c_{k,\gamma} := x(\gamma_k) - x\left(\gamma_k^C\right)$, and write $c_k := c_{k,\gamma}$. Then, for $k \geq 2$, we have the relations*

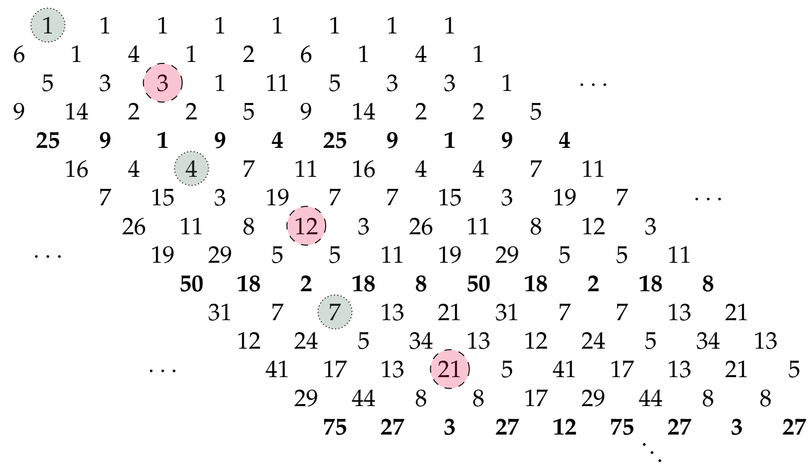


Figure 17: Arithmetic progressions in a frieze of type D.

(1) $c_k = (s_{k-1} - s_{k-2})c_1 + c_{k-2}$, *where we define* $c_0 = c_1$;
(2) $c_k = c_1 \left(1 + \sum_{i=0}^{k-1} (-1)^{i+\alpha} s_i\right)$, *where* $\alpha = 1$ *if* k *is even and* $\alpha = 0$ *otherwise.*

Note that, if $i = j$, then $c_1 = x(\gamma_1)$.

3.5 Arithmetic progressions

Tschabold showed that each diagonal of a frieze (of positive integers) arising from a once-punctured disk is made up of a collection of arithmetic progressions [21, Prop. 3.11]. The dotted and dashed circles in Figure 17 highlight two such arithmetic progressions.

Proposition 3.8 (Analog of [21, Thm. 3.11])**.** *Suppose* (S, M) *is a once-punctured disk. Let* $\gamma_1 = \gamma$ *be an ordinary arc from* i *to* j *(possibly* $i = j$*) or a boundary edge from* i *to* $i + 1$*. Then, for* $k \geq 2$*, we have* $x(\gamma_k) = x(\gamma_{k-1}) + \left(x(\gamma_1) + x(\gamma_1^C) \right)$.

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